

# PHANTOM MAPS AND FORGETABLE MAPS

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**ABSTRACT.** In this note, we attack a question posed ten years ago by Tsukiyama about the injectivity of the so- called Forgetable map. We show that we can insert the Forgetable map in an exact sequence and that the problem can be reduced to the computation of the sequence which turns out unexpectedly to be related to the phantom map problem and the famous Halperin conjecture in rational homotopy theory.

## 1. INTRODUCTION

Homotopy theory is now an area which contains many research topics. Among them are two interesting ones : Phantom map theory which is well known and Forgetable map which is maybe not so well known. An interesting relation have been observed in this paper which is used to attack the Forgetable map problem posed by Tsukiyama[28],[29] ten years ago.

A pair of maps  $f$  and  $g$  from a CW complex  $X$  to a topological space  $Y$  is called a phantom pair if the restriction of  $f$  and  $g$  to the  $n$ -skeleton of the complex  $X$  is homotopic for all  $n \geq 0$ . In this case , we call the map  $f$  a  $g$ - phantom map with respect to the map  $g$ . Denote by  $Ph^g(X, Y)$  the set of homotopy classes of  $g$ -phantom maps from  $X$  to  $Y$ . It is clear that the concept of  $g$ -phantom map is homotopy invariant. Especially , if  $g = \text{constant map}$ , then the  $g$ -phantom map was called just a phantom map and if  $g = id$ , then  $f$  is a  $g$ -phantom map if and only if  $f$  is a weak identity as defined by Roitberg[22].

Historically Adams and Walker[1] found the first nontrivial phantom map and Gray made the first detailed study of phantom maps in his PhD thesis[9] . Later many other authors , McGibbon, Meier , Moller

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, Oda-Shitanda , Roitberg, Sullivan, Zabrodsky , etc. contributed a lot of ideas to this beautiful area, see [15] for a comprehensive survey of this area. Among many others, they proved the following which are crucial to our later application .

**Theorem 1.1.** [15] *Let  $X$  and  $Y$  be nilpotent CW complexes with finite type. Then the set  $Ph^g(X, Y)$  is either one point set or uncountable.*

In general,  $Ph^*(X, Y)$  is a proper subset of  $[X, Y]$  . But the following theorem says they are equal under some conditions and even more is true.

**Theorem 1.2.** [15] *Let  $X$  and  $Y$  be nilpotent CW complexes with finite type . If  $[X_\tau, Y] = [\Sigma X_\tau, Y] = *$ , then*

$$(1.1) \quad Ph^*(X, Y) = [X, Y] = [X_{(0)}, Y] = \prod_{n \geq 0} H^n(X, \pi_{n+1}(Y) \otimes R)$$

**Theorem 1.3.** [15] *Let  $Y = \Omega^n K$  and  $X = \Sigma^m Z$  for  $m, n \geq 0$  where  $K$  is a simply connected finite CW complex .If  $Z$  satisfies one of the following conditions, then we have*

$$(1.2) \quad Ph^*(X, Y) = [X, Y] = [X_{(0)}, Y] = \prod_{n \geq 0} H^n(X, \pi_{n+1}(Y) \otimes R)$$

- $Z = BG, G$  is a connect compact lie group with finite fundamental group ,or
- $Z$  is a connected infinite loop space with finite fundamental group ,or
- $Z$  is a 1-connected finite Postnikov space, i.e.  $\pi_i Z = 0$  if  $i$  is large enough.

In the above theorem the target space is the (iterated) loop space of a finite CW complex. To deal with some essential infinite space Zabrodsky extend the above theorem as follows

**Theorem 1.4.** [30] *The equation (1.1) remains true if  $X = \Sigma^m K(H, l+2)$  and  $Y = \Omega^n BAut(K)$  where  $m, l, n \geq 0$ ,  $H$  is an abelian group and  $K$  is a finite CW complex.*

Theorem 1.3 and 1.4 say, in some case ,all maps are phantom maps and the homotopy classes of them can be calculated . The general phantom pair is studied only briefly by Oda-Shitanda[20] and seems to be forgot later. Roitberg[22, 23] has studied the weak identities and posed several interesting open questions about them and later Shitanda had also some related works on it.

According to our point of view the main problem one faces with the general phantom pair is the following

**Question 1.5.** *Let  $g_1, g_2 : X \rightarrow Y$  are two maps. What is the implication relation between  $Ph^{g_1}(X, Y) = \{g_1\}$  and  $Ph^{g_2}(X, Y) = \{g_2\}$*

A well known result in this direction is

**Theorem 1.6.** [20] *If  $Y$  is a H-space with inverse, then for any two maps  $g_1, g_2 : X \rightarrow Y$  the equations in Question 1.5 are equivalent.*

In our application we have to extend the Theorem 1.6 to the case where  $Y$  is not a H-space. Actually what we need is about somewhat more general notion. See Theorem 2.13, 2.14 for details.

Now let us turn to Forgettable map. Given a principal  $G$ -bundle  $\pi : P \rightarrow B$ , Let

$$aut^G(P) = \{g|g : P \rightarrow P \text{ is a } G\text{-equivariant homotopy equivalence}\}$$

and

$$aut(P) = \{g|g : P \rightarrow P \text{ is a homotopy equivalence}\}$$

There is a natural map  $f : aut^G(P) \rightarrow aut(P)$ . Let

$$Aut^G(P) = \pi_0(aut^G(P))$$

and

$$Aut(P) = \pi_0(aut(P))$$

Then the map  $f$  induces a map

$$F : Aut^G(P) \rightarrow Aut(P)$$

which is called a Forgettable map by Tsukiyama. The question posed by Tsukiyama in [13] is the following

**Question 1.7.** *Is the forgetting map  $F$  injective?*

In [28, 29], Tsukiyama constructed examples which answers negatively the question 1.7 and gave a sufficient condition which answers positively the question 1.7. His example is the following:

**Example 1.8.** *Given a connected compact Lie group, which is not a torus,  $G$  and the maximal torus  $T$ . There is principal  $G$ -bundle  $G \rightarrow G/T \rightarrow BT$  over  $BT$  which is classified by the natural map  $Bi : BT \rightarrow BG$  where  $i : T \rightarrow G$  is the inclusion of  $T$  into  $G$ . Then  $Aut(G/T)$  is finite and there is an exact sequence*

$$0 \rightarrow \pi_1(\text{map}(BT, BG), Bi) \rightarrow Aut^G(G/T) \rightarrow Aut(BT)$$

*Since  $\pi_1(\text{map}(BT, BG), Bi)$  is uncountable,  $Aut^G(G/T)$  is uncountable and thus the Forgettable map  $F : Aut^G(G/T) \rightarrow Aut(G/T)$  is not injective.*

One of the main results in this paper is the following

**Theorem 1.9.** *Let  $\pi : P \rightarrow B$  be a principal  $G$ -bundle with  $P$  a finite CW complex. Then there is an exact sequence*

$$\pi_1 \text{aut}(P) \rightarrow \pi_1(\text{map}_*(BG, \text{Baut}(P)), c) \rightarrow \text{Aut}^G(P) \xrightarrow{F} \text{Aut}(P)$$

where  $c : BG \rightarrow \text{Baut}(P)$  is determined by the principal bundle.

*Remark 1.10.* In the above theorem, the calculation of the  $\text{Ker} F$  is in some sense equivalent to the calculation of the group

$$\overline{G} =: \pi_1(\text{map}_*(BG, \text{Baut}(P)), c)$$

- $\overline{G} = 0 \Rightarrow \text{Ker} F = 0$
- $\overline{G}$  is uncountable  $\Rightarrow \text{Ker} F$  is uncountable since

$$\pi_1 \text{aut}(P) = \pi_1(\text{map}(P, P), \text{id})$$

is countable.

The following Theorem shows that, in certain case,  $\text{Ker} F$  is either zero or uncountable.

**Theorem 1.11.** *Let  $\pi : P \rightarrow B$  be as above,  $k : B \rightarrow BG$  the classifying map,  $\bar{k} : \bar{B} \rightarrow BG$  the associated fibration with fiber  $\bar{P}$  and  $c : BG \rightarrow \text{Baut}(P)$  the classifying map of  $\bar{k}$ . The followings are true.*

- If  $\text{Ker} F$  is uncountable if  $\text{Ph}_1^i(BG, \text{Baut}(P)) \neq 0$
- If  $c$  is phantom map and  $G$  be a connected compact Lie group,  $\text{Ker} F$  is either zero or uncountable
- If  $G$  be a 1-connected  $K(H, m)$ ,  $\text{Ker} F$  is either zero or uncountable

According to Tsukiyama[28][29], it is possible that  $\text{Ker} F$  is zero or uncountable. The above theorem says in certain case this is the only possibilities. Furthermore we will show that results in phantom map theory and rational calculation which is usually not so difficult can be used to decide when the  $\text{Ker} F$  is zero or uncountable. The theorem above leads to another natural question

**Question 1.12.** *Is it possible that the  $\text{Ker} F$  is finite or countable?*

Now we will give concrete conditions for injectivity or noninjectivity of the Forgetable map. First we assume the group  $G$  to be a connected compact Lie group.

**Theorem 1.13.** *Let  $P$  be a 1-connected CW complex. Then the followings are equivalent:*

- *there is a connected compact Lie group  $G$  and a principal  $G$ -bundle such that the total space has the homotopy type of  $P$ , the classifying map  $c$  is a phantom map and the associated Forgetable map  $F$  has uncountable kernel*
- $\bigoplus_{i>0} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id)$  *is nontrivial*

Before giving concrete examples of principal bundles with noninjective Forgetable map, we recall some backgrounds. A space  $P$  is called elliptic if  $H^i(X, Q) = 0$  and  $\pi_i(X) \otimes Q = 0$  when  $i$  is sufficient large. Let  $(X, *)$  be any pointed space. The Gottlieb group (or evaluation subgroup)[10] is defined by

$$G_n(X) = \text{Im}\{ev_* : \pi_n(\text{map}(X, X); id) \rightarrow \pi_n(X)\}$$

where  $ev : \text{map}(X, X)_{id} \rightarrow X$  is defined by  $ev(f) = f(*)$

The Gottlieb groups are extremely difficult to compute in general. However for rational space there have been some remarkable results on the Gottlieb groups.

**Theorem 1.14.** [7] *If  $X$  is finite CW complex, then  $G_{\text{even}}(X_{(0)}) = 0$*

**Theorem 1.15.** [25] *Let  $X$  be a finite 1-connected complex, then  $G_*(X_{(0)}) \neq 0$  if  $X$  is rationally nontrivial and elliptic.*

Now the following Theorem follows immediately from the last three theorems.

**Theorem 1.16.** *Let  $P$  be any 1-connected elliptic finite CW complex. Then there is a compact Lie group  $G$  and a principal  $G$ -bundle  $\pi : P' \rightarrow B$  such that the Forgetable map has uncountable kernel and  $P, P'$  has the same homotopy type.*

*Remark 1.17.* We don't know if there exists 1-connected CW complex  $P$  such that  $\bigoplus_{i>0} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$ .

If we assume  $G = K(H, 2m)$  where  $H$  is a finitely generated abelian group and  $m \geq 1$ . We have the following

**Theorem 1.18.** *Let  $P$  be a 1-connected CW complex. Then the followings are equivalent:*

- *for all  $m \geq 1$ , finitely generated abelian group  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$ , the associated Forgetable map is injective*
- $\bigoplus_{i>1} \pi_{2i}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$

Now we want to give some examples with injective Forgetable map. Again we first recall some backgrounds. For simplicity we will assume spaces involved are 1-connected.

A space  $X$  is said to be of type  $F_0$  if  $\dim H^*(X; Q) < \infty$ ,  $\dim \pi_*(X) \otimes Q < \infty$  and  $H^{odd}(X; Q) = 0$ . One of the most beautiful conjectures in rational homotopy theory is the following

**Conjecture 1.19.** [6] *Let  $P \rightarrow E \rightarrow B$  be a fibration such that the fiber  $P$  is homotopy equivalent to a CW complex of type  $F_0$ . Then the Serre spectral sequence (with rational coefficients in  $Q$ ) of the fibration collapses at the  $E^2$  term.*

In his two remarkable papers [17, 18], W.Meier found the relation between Halperin conjecture and the vanishing of the  $\pi_{even} \text{map}((P_{(0)}, P_{(0)}); id)$ , i.e.

**Theorem 1.20.** *Let  $P$  be of type  $F_0$ . Then the followings are equivalent.*

- *The Serre spectral sequence of every fibration with fiber  $P$  collapses at the  $E^2$  term*
- $\pi_{even} \text{map}((P_{(0)}, P_{(0)}); id) = 0$

The Halperin conjecture have been verified for a number of special cases. The result obtained so far can be stated as follows

**Theorem 1.21.** [2, 14, 24, 27] *Let  $P$  be a space satisfying one of the following conditions. Then the Halperin conjecture is true:*

- *$P$  is a Kahler manifold*
- *$H^*(P; Q)$  as an algebra has at most 3 generators*
- *$P = G/U$  where  $G$  is a compact Lie group and  $U$  is a closed subgroup of maximal rank*

Comparing Theorem1.18 and Theorem1.20 we obtain immediately the following

**Theorem 1.22.** *Let  $P$  be a 1-connected finite CW complex of type  $F_0$ . Then the followings are equivalent:*

- *The Halperin conjecture is true for  $P$*
- *For every  $m \geq 1$ , finitely generated abelian group  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$ , the associated Forgetable map is injective and  $\pi_2 \text{map}((P_{(0)}, P_{(0)}); id) = 0$*

**Corollary 1.23.** *Let  $P$  be a 1-connected finite CW complex satisfying the condition of Theorem1.21. Then For all  $m \geq 1$ , finitely generated*

*abelian group  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$ , the associated Forgettable map is injective*

In section 2 we will introduce the phantom element which is a generalization of phantom pair and use this concept to prove Theorem 2.13 and Theorem 2.14. In section 3, we will study the forgetting map and try to insert it into an exact sequence and prove the Theorem 1.9. In section 4, the results of previous sections will be applied here to get the precise information about the Forgettable map and prove Theorem 1.11, 1.13, 1.18. In this paper all our basic spaces will be assumed to be CW complexes with finite type. We will also use the following notations:

- $X_n$  is the  $n$ -th skeleton of  $X$
- $map(X, Y)$  is the space of continuous mappings from  $X$  to  $Y$
- $map_*(X, Y)$  is the subspace of pointed mappings from  $(X, x_0)$  to  $(Y, y_0)$
- $l : X \rightarrow X_{(0)}$  is the rationalization
- Let  $\tau : X_\tau \rightarrow X$  be the homotopy fiber of  $l$ . Then  $X_\tau \xrightarrow{\tau} X \rightarrow X_{(0)}$  is a cofibration up to homotopy
- $\hat{e} : Y \rightarrow \hat{Y}$  is the profinite completion of Sullivan [26]

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## 2. PHANTOM ELEMENT

Let us begin with some definitions. Let  $X$  be a CW complex,  $Y$  a space and  $f, g : X \rightarrow Y$  two maps. A map  $f : X \rightarrow Y$  is called a phantom map if  $f|_{X^n}$  (the restriction of  $f$  to the  $n$ -skeleton of  $X$ ) is homotopic to the constant map for all  $n \geq 0$ . In [20], Oda and Shitanda defined that  $f$  and  $g$  are a phantom pair if  $f|_{X^n}$  and  $g|_{X^n}$  are homotopic for all  $n \geq 0$ . For a fixed map  $g : X \rightarrow Y$  we denote by  $Ph^g(X, Y)$  the set of homotopy classes of maps  $f$  such that  $f$  and  $g$  are phantom pair. Each element of  $Ph^g(X, Y)$  is also called a  $g$ -phantom map.

Here we generalize the concept of the phantom pair as follows

**Definition 2.1.** Let spaces  $X$  be a CW complex,  $Y$  be a space and  $g : X \rightarrow Y$  any map. Then an element  $\alpha \in \pi_j(map_*(X, Y); g)$  is called a  $g$ -phantom element if  $(i_n^*)_\#(\alpha) = 0$  for all  $n \geq 0$  where  $(i_n^*)_\# : map_*(X, Y) \rightarrow map_*(X^n, Y)$  is the homomorphism induced by the

inclusion  $i_n : X^n \rightarrow X$ . Denoted by

$$Ph_j^g(X, Y) = \{\alpha \in \pi_j(\text{map}_*(X, Y); g) \mid \alpha \text{ is a } g\text{-phantom element}\}$$

If  $j = 0$ , then  $\alpha$  is a  $g$ -phantom element iff it represents the homotopy class of a map which is a  $g$ -phantom map. If  $g = \text{constant map}$ , then a  $g$ -phantom map is the same as phantom map. Since Adams and Walker[1] found the first essential phantom map, this area has attracted interests of many mathematicians.

Let us recall some basic results about homotopy of a sequence of fibrations at first. Let

$$\cdots \rightarrow X_n \xrightarrow{\pi_n} X_{n-1} \rightarrow \cdots$$

be a sequence of fibrations of spaces and  $X$  be the inverse limit of the above inverse system. If we choose base points  $x_n \in X_n$  such that  $\pi_n(x_n) = x_{n-1}$ . It was shown by Bousfield and Kan in [5] that there exists the following short exact sequence for  $j \geq 0$

$$* \rightarrow \lim_{\leftarrow n}^1 \pi_{j+1}(X_n, x_n) \rightarrow \pi_j(\lim_{\leftarrow} X_n, x_n) \rightarrow \lim_{\leftarrow} \pi_j(X_n, x_n) \rightarrow *$$

Now if  $X$  is a CW complex with skeleton  $X^n$  and  $Y$  a space, then

$$\cdots \rightarrow \text{map}_*(X^n, Y) \xrightarrow{i_{n,n-1}^*} \text{map}_*(X^{n-1}, Y) \rightarrow \cdots$$

is a sequence of fibrations with  $\text{map}_*(X, Y)$  as the inverse limit.

**Corollary 2.2.** *Let  $X$  and  $Y$  be nilpotent CW complexes of finite type and  $g : X \rightarrow Y$  be any map. Then for all  $j \geq 0$  there exists a short exact sequence*

$$\begin{aligned} * \rightarrow \lim_{\leftarrow n}^1 \pi_{j+1}(\text{map}_*(X^n, Y); g|X^n) &\rightarrow \pi_j(\text{map}_*(X, Y); g) \rightarrow \\ \lim_{\leftarrow n} \pi_j(\text{map}_*(X^n, Y); g|X^n) &\rightarrow * \end{aligned}$$

**Corollary 2.3.** *Let  $X$  and  $Y$  be nilpotent CW complexes of finite type and  $g : X \rightarrow Y$  be any map. Then for all  $j \geq 0$ , we have*

$$Ph_j^g(X, Y) = \lim_{\leftarrow n}^1 \pi_{j+1}(\text{map}_*(X^n, Y); g|X^n)$$

If  $j = 0$ , we recover  $Ph^g(X, Y) = \lim_{\leftarrow n}^1 \pi_1(\text{map}_*(X^n, Y); g|X^n)$

A natural problem about  $Ph_j^g(X, Y)$  is its cardinality. For this we have the following

**Theorem 2.4.** [15] *The first derived inverse limit of an inverse system of countable groups is either one point set or uncountable.*



**Corollary 2.5.** *Let  $X$  and  $Y$  be nilpotent CW complexes of finite type and  $g : X \rightarrow Y$  be any map. Then  $Ph_j^g(X, Y)$  is either one point set or uncountable for all  $j \geq 0$ .*

Another natural question is the extended version of Question 1.5.

**Question 2.6.** *For two maps  $f, g : X \rightarrow Y$ , what is the relation between  $Ph_j^g(X, Y)$  and  $Ph_j^f(X, Y)$ ?*

The first result in this direction is an extension of the result of Oda-Shitanda[20].

**Theorem 2.7.** *Let  $X$  be a CW complex and  $Y$  be an H-space with inverse. Then we have  $Ph_j^g(X, Y) = Ph_j^f(X, Y)$*

*Proof.* If  $j = 0$ , this is Theorem 3.7(2) of Oda-Shitanda[20] and if  $j > 0$ , this follows from the following lemma and the Corollary 2.3.  $\square$

**Lemma 2.8.** *Let  $i : X_1 \rightarrow X_2$  be a map and  $Y$  be an H-space with inverse. Then for any map  $f : X_2 \rightarrow Y$  the following diagram is commutative up to homotopy*

$$\begin{array}{ccc} map_*(X_2, Y)_f & \xrightarrow{i^*} & map_*(X_1, Y)_{f \circ i} \\ \downarrow h_2 & & \downarrow h_1 \\ map_*(X_2, Y)_* & \xrightarrow{i^*} & map_*(X_1, Y)_* \end{array}$$

where  $h_1, h_2$  are defined during the proof.

*Proof.* Since  $Y$  is an H-space with inverse, there is a multiplication  $\mu : Y \times Y \rightarrow Y$ , a map  $\nu : Y \rightarrow Y$  is called an inverse for  $\mu$  if each composite map

$$Y \xrightarrow{(1, \nu)} Y \times Y \xrightarrow{\mu} Y \text{ and } Y \xrightarrow{(\nu, 1)} Y \times Y \xrightarrow{\mu} Y$$

are homotopic to the constant map  $*$  :  $Y \rightarrow Y$ . From these we can construct the maps  $h_1, h_2$  by the composite

$$\begin{aligned} map_*(X_j, Y)_{f_j} &\xrightarrow{(1, \nu_*)} map_*(X_j, Y)_{(f_j \times map_*(X_j, Y)_{\nu \circ f_j})} \rightarrow \\ &\rightarrow map_*(X_j, Y \times Y)_{(f_j, \nu \circ f_j)} \xrightarrow{\mu_*} map_*(X_j, Y)_* \end{aligned}$$

where  $f_2 = f$  and  $f_1 = f \circ i$

It is easy to see from the definition of the maps  $h_1, h_2$  that the diagram in the lemma is commutative up to homotopy.  $\square$

**Theorem 2.9.** [21] *Let  $X, Y$  be nilpotent CW complexes of finite type and  $f : X \rightarrow Y$  be any map. Then  $\alpha \in \pi_j(\text{map}_*(X, Y); f)$  is a phantom element  $\Leftrightarrow$*

- $(\hat{e}_*)_{\#}(\alpha) = 0$  where  $(\hat{e}_*)_{\#} : \pi_j(\text{map}_*(X, Y); f) \rightarrow \pi_j(\text{map}_*(X, \hat{Y}); \hat{f})$
- $\tau_{\#}^*(\alpha) = 0$  where  $\tau_{\#}^* : \pi_j(\text{map}_*(X, Y); f) \rightarrow \pi_j(\text{map}_*(X_{\tau}, Y); f_{\tau})$

*Proof.* We will only prove the "if" part which is necessary for our application in this paper. For the proof of the other parts, see [21].

Let  $\alpha \in \pi_j(\text{map}_*(X, Y); f)$  such that  $(\hat{e}_*)_{\#}(\alpha) = 0$ . If we consider the following commutative diagram

$$\begin{array}{ccc} \pi_j(\text{map}_*(X, Y); f) & \xrightarrow{(i_n^*)_{\#}} & \pi_j(\text{map}_*(X^n, Y); f|_{X^n}) \\ \downarrow (\hat{e}_*)_{\#} & & \downarrow (\hat{e}_*)_{\#} \\ \pi_j(\text{map}_*(X, \hat{Y}); \hat{f}) & \xrightarrow{(i_n^*)_{\#}} & \pi_j(\text{map}_*(X^n, \hat{Y}); \hat{f}|_{X^n}) \end{array}$$

then we have  $(\hat{e}_*)_{\#} \circ (i_n^*)_{\#}(\alpha) = (i_n^*)_{\#} \circ (\hat{e}_*)_{\#}(\alpha) = 0$ . In [26], Sullivan showed that if  $Y$  is a nilpotent space,  $\hat{e} : Y \rightarrow \hat{Y}$  and  $h, g : Z \rightarrow Y$  be any two maps where  $Z$  any finite CW complex such that  $\hat{e} \circ g \simeq \hat{e} \circ h$ , then  $g \simeq h$ .

By the result of Sullivan, it follows immediately that the map

$$\hat{e}_* : \text{map}_*(X^n, Y)_{f|_{X^n}} \rightarrow \text{map}_*(X^n, \hat{Y})_{\hat{f}|_{X^n}}$$

has also the property above. Thus the induced homomorphism

$$(\hat{e}_*)_{\#} : \pi_j(\text{map}_*(X^n, Y); f|_{X^n}) \rightarrow \pi_j(\text{map}_*(X^n, \hat{Y}); \hat{f}|_{X^n})$$

is injective. This completes the proof.

Next we show that if  $(\tau^*)_{\#}(\alpha) = 0$ , then  $\alpha \in \pi_j(\text{map}_*(X, Y); f)$  is a phantom element. From the assumption, we have

$$(\hat{e}_*)_{\#}(\tau^*)_{\#}(\alpha) = 0 \in \pi_j(\text{map}_*(X_{\tau}, \hat{Y}); \hat{f}_{\tau})$$

By Proposition 2.1 of [20],  $\text{map}_*(X_{(0)}, \hat{Y})$  is weakly contractible and hence the natural map  $\pi_j(\text{map}_*(X, \hat{Y}); \hat{f}) \rightarrow \pi_j(\text{map}_*(X_{\tau}, \hat{Y}); \hat{f}_{\tau})$  is an isomorphism for  $j > 0$ . Since  $(\hat{e}_*)_{\#} \circ (\tau^*)_{\#} = (\tau^*)_{\#} \circ (\hat{e}_*)_{\#}$ , we have

$$\hat{\alpha} = 0 \in \pi_j(\text{map}_*(X, \hat{Y}); \hat{f})$$

By the first part,  $\alpha$  is a phantom element. □

**Proposition 2.10.** *Let  $X, Y$  be CW complexes such that  $[\Sigma^j X_{\tau}, Y] = 0$  and  $[\Sigma^{j+1} X_{\tau}, Y] = 0$ . If  $f : X \rightarrow Y$  is a phantom map, then we have*

$$Ph_j^f(X, Y) = \pi_j(\text{map}_*(X, Y); f) = \pi_j(\text{map}_*(X_{(0)}, Y); f_{(0)})$$

*Proof.* To show the first equation, it suffices to prove by Theorem 2.9 that

$$\pi_j(\text{map}_*(X_\tau, Y); f_\tau) = 0$$

From Theorem 5.1 of [15] we have  $f_\tau \simeq 0$  and this implies

$$\pi_j(\text{map}_*(X_\tau, Y); f_\tau) \cong \pi_j(\text{map}_*(X_\tau, Y); *) \cong [\Sigma^j X_\tau, Y] = 0$$

which is what we want to prove.

Similarly we can prove  $\pi_{j+1}(\text{map}_*(X_\tau, Y); f_\tau) = 0$ . By using the fibration

$$\text{map}_*(X_{(0)}, Y) \rightarrow \text{map}_*(X, Y) \rightarrow \text{map}_*(X_\tau, Y)$$

and the fact that

$$\pi_j(\text{map}_*(X_\tau, Y); f_\tau) = \pi_{j+1}(\text{map}_*(X_\tau, Y); f_\tau) = 0$$

we obtain immediately the second equation.  $\square$

**Corollary 2.11.** *Let  $X, Y, f$  be as in Proposition 2.10. If we assume further that  $Y$  is a 1-connected rational H-space. Then*

$$Ph_j^f(X, Y) = \pi_j(\text{map}_*(X, Y); f) = [\Sigma^j X_{(0)}, Y]$$

Before the proof of the Corollary let us first recall some useful results.

**Lemma 2.12.** [30] *Let  $X, Y_1, Y_2$  be 1-connected CW complexes with finite type and  $t : Y_1 \rightarrow Y_2$  be a rational equivalence. Then the induced map  $t_* : \text{map}_*(X_{(0)}, Y_1) \rightarrow \text{map}_*(X_{(0)}, Y_2)$  is a weak equivalence.*

*Proof of Corollary 2.11.* Let  $t : Y \rightarrow \bar{Y}$  be an integral approximation of  $Y$  (see [30] for the concept of integral approximation of a space). Then by the above Lemma, the map  $t_* : \text{map}_*(X_{(0)}, Y) \rightarrow \text{map}_*(X_{(0)}, \bar{Y})$  is a weak equivalence. Since  $Y$  is a rational H-space, we can choose  $\bar{Y}$  to be an H-space (see remark by McGibbon in [15]). It is well known that the different components of a mapping space  $\text{map}_*(X_{(0)}, \bar{Y})$  have the same homotopy type if  $\bar{Y}$  is an H-space. This completes the proof.  $\square$

**Theorem 2.13.** *Let  $X = K(H, m+1)$ ,  $Y = \text{Baut}(P)$  and  $f : X \rightarrow Y$  is any map where  $P$  is a simply connected finite CW complex and  $m \geq 2$ . Then*

$$Ph_j^f(X, Y) = \pi_j(\text{map}_*(X, Y); f) = [\Sigma^j X_{(0)}, Y]$$

*Proof.* We know from Corollary C' of [30] that  $\text{map}_*(X_\tau, Y)$  is weakly contractible. It follows that any map  $f : X \rightarrow Y$  is a phantom map. Thus we can apply the Proposition 2.10 to get

$$Ph_j^f(X, Y) = \pi_j(\text{map}_*(X, Y); f) = \pi_j(\text{map}_*(X_{(0)}, Y); f_{(0)})$$

To prove the last equation, note that, if  $X$  is 1-connected, then we have

$$\text{map}_*(X, Y) \simeq \text{map}_*(X, \tilde{Y})$$

where  $\tilde{Y}$  is the universal covering of  $Y$ . Now  $Y$  and thus  $\tilde{Y}$  is a 1-connected rational H-space. It follows from Corollary 2.11 and the above equivalence that

$$\pi_j(\text{map}_*(X_{(0)}, Y); f_{(0)}) = \pi_j(\text{map}_*(X_{(0)}, Y); *)$$

□

**Theorem 2.14.** *Let  $X = BG$ ,  $Y = \text{Baut}(P)$  and  $f : X \rightarrow Y$  is a phantom map where  $G$  is a connected compact Lie group and  $P$  is 1-connected finite CW complex. Then for  $j \geq 1$  we have*

$$Ph_j^f(X, Y) = \pi_j(\text{map}_*(X, Y); f) = [\Sigma^j X_{(0)}, Y]$$

*Proof.* By Proposition 2.10, to prove the first equation it suffices to prove that  $[\Sigma^j X_\tau, Y] = [\Sigma j + 1 X_\tau, Y] = 0$ . Now for  $i \geq 1$ , we have

$$\begin{aligned} [\Sigma^i BG_\tau, Y] &= [\Sigma^{i-1} BG_\tau, \Omega Y] = \\ &= [\Sigma^{i-1} BG_\tau, \text{aut}(P)] = 0 \end{aligned}$$

where the last equation follows from Theorem C(c) of Zabrodsky [30].

The proof of the second equation follows from the same argument as in the proof of Theorem 2.13. □

### 3. FORGETABLE MAP AND ITS DESCRIPTION

Let us consider the principal  $G$ -bundle  $q : P \rightarrow B$  with structure group  $G$  where  $G$  acts on  $P$  freely. For each such bundle one can consider the space  $\text{aut}^G(P)$  of unbased  $G$ -equivariant self-homotopy equivalences of  $P$  and the group

$$\text{Aut}^G(P) = \pi_0(\text{aut}^G(P))$$

which is called the group of  $G$ -equivariant self-homotopy equivalences of the principal bundle  $q : P \rightarrow B$ . On the other hand, we can also consider the space  $\text{aut}(P)$  of unbased self-homotopy equivalences of space  $P$  and the group

$$\text{Aut}(P) = \pi_0(\text{aut}(P))$$

which is called the group of unbased self-homotopy equivalences of  $P$ . There have been extensive study on these two subjects, see [13] and the extensive references there. About ten years ago, Tsukiyama [13] posed the following

**Question 3.1.** *When is the natural map  $F : \text{Aut}^G(P) \rightarrow \text{Aut}(P)$ , which forgets the  $G$ -action, a monomorphism?*

No progress has been made unless Tsukiyama's two recent papers [28] [29]. In this paper, we will try to attack this question. Our approach is based on the identification of the space of  $G$ -equivariant self-homotopy equivalences as the loop space on a mapping space and the recent results on the Sullivan conjecture. What is most interesting about our results is the relation between the injectivity of  $F$  and the existence of the phantom map between appropriate spaces.

In [28] [29], Tsukiyama used an indirect approach to attack the question and got partial results on it. In this section, based on a simple but crucial observation, we will identify the homomorphism  $F$  as the homomorphism induced on  $\pi_0$  by a map whose homotopy fiber can be determined explicitly and thus can determine the kernel of  $F$  under reasonable condition. Now let  $G$  be a topological group,  $q : P \rightarrow B$  be a principal  $G$ -bundle and  $k : B \rightarrow BG$  be its classifying map. For the map  $k$ , we can take  $\bar{k} : \bar{B} \rightarrow BG$  as a fibration via the standard factorization of a map into the composite of a homotopy equivalence and a Hurewicz fibration. Given fibration  $\bar{k} : \bar{B} \rightarrow BG$ , we can form the group  $\text{Aut}_{BG}(\bar{B})$  the group of homotopy classes of self homotopy equivalences of  $\bar{B}$  over  $BG$ . The following is a well known result [4],[11],[12].

**Proposition 3.2.** *There is a natural isomorphism*

$$\text{Aut}^G(P) \cong \text{Aut}_{BG}(\bar{B}) \cong \pi_1(\text{Map}(BG, \text{Baut}(P)), c)$$

where map  $c : BG \rightarrow \text{Baut}(P)$  is determined by the principal bundle.

If the above isomorphism is natural in object  $G$ , then the map  $F$  will be naturally isomorphic to the map

$$ev_* : \pi_1(\text{Map}(BG, \text{Baut}(P)), c) \rightarrow \pi_1 \text{Baut}(P)$$

whose kernel can be computed explicitly by the evaluation fibration

$$\text{Map}_*(BG, \text{Baut}(P))_c \rightarrow \text{Map}(BG, \text{Baut}(P))_c \rightarrow \text{Baut}(P)$$

A careful check confirms the above speculation and leads to the following which is the Theorem 1.9 in the Introduction.

**Theorem 3.3.** *Let  $q : P \rightarrow B$  be a principal  $G$ -bundle,  $c : BG \rightarrow \text{Baut}(P)$  be the classifying map for fibration  $\bar{k} : \bar{B} \rightarrow BG$ . Then there*

is a commutative diagram

$$\begin{array}{ccc} \pi_0 \text{aut}_{BG}(\bar{B}) & \xrightarrow{\cong} & \pi_0 \text{aut}^G(P) \\ \downarrow & & \downarrow \\ \pi_0 \text{aut}(\bar{P}) & \xrightarrow{\cong} & \pi_0 \text{aut}(P) \end{array}$$

where  $\bar{P}$  is the fiber of the fibration  $\bar{k}$  which is homotopy equivalent to  $P$  and the two horizontal maps are isomorphisms. It follows from the diagram above immediately that the following sequence is exact

$$\pi_1 \text{aut}(P) \rightarrow \pi_1(\text{Map}_*(BG, \text{Baut}(P)), c) \rightarrow \text{Aut}^G(P) \rightarrow \text{Aut}(P)$$

This theorem follows directly from the following lemmas. Let  $q : P \rightarrow B$  be a principal  $G$ -bundle and  $k : X \rightarrow B$  be a map, then there is an associated principal  $G$ -bundle over  $X$  defined by

$$\begin{array}{ccc} k^*(P) & \longrightarrow & P \\ \downarrow k^*(q) & & \downarrow q \\ X & \xrightarrow{k} & B \end{array}$$

**Lemma 3.4.** *Let  $\pi : EG \rightarrow BG$  be the universal principal  $G$ -bundle, then the rule that takes  $k$  to  $k^*(\pi)$  defines a natural bijection from  $[B, BG]$ , the set of free homotopy classes of maps from  $B$  to  $BG$ , to the set of isomorphism classes of  $G$ -bundles over  $B$ .*

*Proof.* well known. □

**Lemma 3.5.** *If  $q : P \rightarrow B$  is a principal  $G$ -bundle and  $g : X \rightarrow B$  is a homotopy Equivalence, then the induced bundle map from  $g^*(q) : g^*(P) \rightarrow X$  to  $q$  is a homotopy equivalence between two principal bundles.*

*Proof.* This is (1.9) of [3]. □

**Lemma 3.6.** *Let  $q : P \rightarrow B$  and  $k : B \rightarrow BG$  be as above. Then there is a commutative diagram .*

$$\begin{array}{ccc} \text{aut}^G(k^*(EG)) & \longrightarrow & \text{aut}^G(P) \\ \downarrow & & \downarrow \\ \text{aut}(k^*(EG)) & \longrightarrow & \text{aut}(P) \end{array}$$

Where the two vertical maps are Forgetable maps and the horizontal maps are homotopy equivalences and are defined in the proof.

*Proof.* By Lemma 3.4, There is a principal bundle isomorphism over  $B$   $h : P \rightarrow k^*(EG)$ . Define the horizontal maps by the rule

$$x \longmapsto h^{-1} \circ x \circ h$$

It is obvious that the diagram is commutative.  $\square$

**Lemma 3.7.** *Let  $q : P \rightarrow B$ ,  $k : B \rightarrow BG$  and  $\bar{k} : \bar{B} \rightarrow BG$  be as above. Then there is a commutative diagram up to homotopy.*

$$\begin{array}{ccc} \text{aut}^G(\bar{k}^*(EG)) & \longrightarrow & \text{aut}^G(k^*(EG)) \\ \downarrow & & \downarrow \\ \text{aut}(\bar{k}^*(EG)) & \longrightarrow & \text{aut}(k^*(EG)) \end{array}$$

Where the two vertical maps are Forgetable maps and the horizontal maps are homotopy equivalences and are defined in the proof.

*Proof.* By Lemma 3.5, There is a homotopy equivalence  $h$  of two principal bundles  $k^*(\pi) : k^*(EG) \rightarrow B$  and  $\bar{k}^*(\pi) : \bar{k}^*(EG) \rightarrow \bar{B}$ . As in the proof of the lemma above, define the horizontal maps similarly. Then the diagram is easily seen to be commutative up to homotopy.  $\square$

**Lemma 3.8.** *Let  $q : P \rightarrow B$ ,  $k : B \rightarrow BG$  and  $\bar{k} : \bar{B} \rightarrow BG$  be as above. Then there is a commutative diagram up to homotopy.*

$$\begin{array}{ccc} \pi_0 \text{aut}_{BG}(\bar{B}) & \longrightarrow & \pi_0 \text{aut}^G(\bar{k}^*(EG)) \\ \downarrow & & \downarrow \\ \pi_0 \text{aut}(\bar{P}) & \longrightarrow & \pi_0 \text{aut}(\bar{k}^*(EG)) \end{array}$$

Where the  $\bar{P}$  is the fiber of the fibration  $\bar{k}$  which is homotopy equivalent to  $P$ , the right vertical map is forgetting maps, the left vertical map is the map by taking a  $f$  to the map which induced on the fiber of the fibration  $k$  at a based point of  $BG$  and the horizontal maps are defined in the proof.

*Proof.* Consider the following diagram

$$\begin{array}{ccccc}
 & \bar{P} & \xrightarrow{f} & \bar{k}^*(EG) & \longrightarrow & EG \\
 & \searrow \tilde{h} & & \downarrow & & \downarrow \\
 & & & \bar{B} & \xrightarrow{=} & BG \\
 & \nearrow \tilde{h} & & \downarrow & & \downarrow \\
 \bar{P} & \xrightarrow{f} & \bar{k}^*(EG) & \xrightarrow{h} & EG & \\
 & \searrow & \downarrow & \nearrow & \downarrow & \\
 & & \bar{B} & \xrightarrow{=} & BG & 
 \end{array}$$

By definition

$$\bar{k}^*(EG) = \{(b, e) \in \bar{B} \times EG \mid \bar{k}(b) = \pi(e)\}$$

$$\bar{P} = \{(b, *) \in \bar{B} \times EG \mid \bar{k}(b) = *\}$$

Now there is an obvious map

$$f : \bar{P} \rightarrow \bar{k}^*(EG), (b, *) \mapsto (b, *)$$

which is a homotopy equivalence by the general property of pullback.

Given a self homotopy equivalence  $h \in \text{aut}_{BG}(\bar{B})$  there exists a map

$$\bar{h} : \bar{k}^*(EG) \rightarrow \bar{k}^*(EG)$$

defined by  $\bar{h}(b, e) = (h(b), e)$  which makes the pair  $(\bar{h}, h)$  a principal bundle map. The given map  $h$  induces also an obvious map  $\tilde{h} \in \text{aut}(\bar{P})$  defined by

$$\tilde{h}(b, *) = (h(b), *)$$

It is easy to check that  $\bar{h} \circ f = f \circ \tilde{h}$

If we define the horizontal maps in the diagram by

$$\pi_0 \text{aut}_{BG}(\bar{B}) \rightarrow \pi_0 \text{aut}^G(\bar{k}^*(EG)), h \mapsto (\bar{h}, h)$$

$$\pi_0 \text{aut}(\bar{P}) \rightarrow \pi_0 \text{aut}(\bar{k}^*(EG)), g \mapsto f \circ g \circ f^{-1}$$

then it is easy to check that the diagram is commutative which is what want to prove.  $\square$



## 4. APPLICATIONS TO THE PROBLEM OF FORGETABLE MAPS

In the last section we have embedded the Forgettable map into the exact sequence

$$\pi_1 aut(P) \rightarrow \pi_1(Map_*(BG, Baut(P)), c) \rightarrow Aut^G(P) \rightarrow Aut(P)$$

In this section we will apply the phantom map theory to extract information about the Forgettable map. If

$$\pi_1(Map_*(BG, Baut(P)), c) = 0$$

then we know that  $Ker F = 0$  from the above exact sequence. If

$$\pi_1(Map_*(BG, Baut(P)), c) \neq 0$$

then we can't say anything about the  $Ker F$ . On the another hand  $\pi_1 aut(P)$  is a countable group if  $P$  is a finite CW complex. It follows that  $Ker F$  is uncountable if

$$\pi_1(Map_*(BG, Baut(P)), c)$$

is uncountable. This is the point where we find the relation between phantom map theory and Forgettable map. From the discussion above, we have the following which is Theorem 1.11 in the Introduction.

**Theorem 4.1.** *Let  $\pi : P \rightarrow B$  be as above,  $k : B \rightarrow BG$  the classifying map,  $\bar{k} : \bar{B} \rightarrow BG$  the associated fibration with fiber  $\bar{P}$  and  $c : BG \rightarrow Baut(P)$  the classifying map of  $\bar{k}$ . The following are true.*

- *If  $Ker F$  is uncountable if  $Ph_1^c(BG, Baut(P)) \neq 0$*
- *If  $c$  is phantom map and  $G$  be a connected compact Lie group,  $Ker F$  is either zero or uncountable*
- *If  $G = K(H, m)$  where  $m \geq 2$  and  $H$  is a finitely generated abelian group,  $Ker F$  is either zero or uncountable*

*Proof.* It suffices to prove the last two statements.

By Theorem 2.13 when  $G$  be a 1-connected  $K(H, m)$  or Theorem 2.14 when  $c$  is phantom map and  $G$  be a connected compact Lie group, we have

$$Ph_1^c(BG, Baut(P)) = \pi_1(map_*(BG, Baut(P)); c)$$

Thus  $\pi_1(map_*(BG, Baut(P)); c)$  is either zero or uncountable □

**Corollary 4.2.** *If the map  $c$  in the Theorem 4.1 is a phantom map, then  $Ker F$  is uncountable iff  $[BG_{(0)}, aut(P)]$  is nontrivial.*

*Proof.* By the Theorem 2.14 for  $j = 1$ , we have

$$Ph_j^c(BG, Baut(P)) = \pi_j(\text{map}_*(BG, Baut(P)); c)$$

Thus the kernel of the Forgetable map is either zero or uncountable. It is zero iff

$$\pi_j(\text{map}_*(BG, Baut(P)); c) = 0$$

By the same Theorem again for  $j = 1$ , we have

$$\pi_1(\text{map}_*(BG, Baut(P)); c) = [\Sigma^1 BG_{(0)}, Baut(P)] = [BG_{(0)}, aut(P)]$$

It follows that kernel of Forgetable map is zero iff

$$[BG_{(0)}, aut(P)] = 0$$

□

The following corollary is Theorem 1.13 in the Introduction.

**Corollary 4.3.** *Let  $P$  be a 1-connected CW complex. Then there is a connected compact Lie group  $G$  and a principal  $G$ -bundle such that the total space has the homotopy type of  $P$ , the classifying map  $c$  is a phantom map and the associated Forgetable map  $F$  has uncountable kernel iff  $\bigoplus_{i>0} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id)$  is nontrivial.*

*Proof.* " $\Leftarrow$ ": First note that given map  $c_0 : BG \rightarrow Baut(P)$  there is a principal bundle such that the total space has the homotopy type of  $P$  and the natural associated map  $c$  is homotopy to the given  $c_0$ .

To prove the Corollary it is sufficient to take  $c_0 = *$  and choose a Lie group  $G$  such that  $[BG_{(0)}, aut(P)] = [BG_{(0)}, \text{map}(P, P)_{id}] \neq 0$ .

According to Theorem 1.2, we have for some even integer  $t > 0$ ,  $[BG_{(0)}, \text{map}(P, P)_{id}] \neq 0$  if  $H^t(BG, Q) \neq 0$  and  $\pi_{t+1}(\text{map}(P_{(0)}, P_{(0)}); id) \neq 0$

Let  $t_0$  be the smallest positive even integer such that

$$\pi_{t_0+1}(\text{map}(P_{(0)}, P_{(0)}); id) \neq 0$$

There exists of course a compact Lie group  $G$  such that  $H^{t_0}(BG, Q) \neq 0$ .

It follows from the discussion above that there exists principal  $G$ -bundle such that the total space has the homotopy type of  $P$  and the associated Forgetable map  $F$  has uncountable kernel.

" $\Rightarrow$ ": If  $\bigoplus_{i>0} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$ , it is easy to see that  $[BG_{(0)}, aut(P)] = 0$  for any connected compact Lie group. This completes the proof by the Corollary 4.2

**Corollary 4.4.** *Let  $P$  be a 1-connected finite CW complex. Then for all principle  $G$ -bundles  $q : P \rightarrow B$  such that the structure group is a connected compact Lie group and the associated map  $c$  is phantom, the associated Forgetable map is injective iff  $\bigoplus_{i>0} \pi_{2i+1}(\text{map}_*(P_{(0)}, P_{(0)}); id) = 0$*

□

Similarly to th Corollary 4.3 we have the following when  $G = K(H, m)$

**Corollary 4.5.** *Let  $P$  be a 1-connected CW complex . Then there is some principal  $K(H, 2m + 1)$  (respectively,  $K(H, 2m)$ )-bundle,  $m \geq 1$ , such that the total space has the homotopy type of  $P$  and the associated Forgetable map  $F$  has uncountable kernel iff  $\bigoplus_{i>1} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id)$  (respectively ,  $\bigoplus_{i>1} \pi_{2i}(\text{map}(P_{(0)}, P_{(0)}); id)$ ) is nontrivial.*

**Corollary 4.6.** *Let  $P$  be a 1-connected CW complex . Then for all  $m \geq 1$  , finitely generated abelian group  $H$  and every principal  $K(H, 2m + 1)$  -bundle with total space homotopy equivalent to  $P$  , the associated Forgetable map is injective iff  $\bigoplus_{i>1} \pi_{2i+1}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$  .*

The following Corollary is the Theorem 1.18 in the Introduction

**Corollary 4.7.** *Let  $P$  be a 1-connected CW complex . Then for all  $m \geq 1$  , finitely generated abelian group  $H$  and every principal  $K(H, 2m)$ -bundle with total space homotopy equivalent to  $P$  , the associated Forgetable map is injective iff  $\bigoplus_{i>1} \pi_{2i}(\text{map}(P_{(0)}, P_{(0)}); id) = 0$  .*

Let us conclude this paper with another question motivated by the results obtained in this paper.

**Question 4.8.** *Is it possible that for every 1-connected CW complex  $P$  there exist a compact Lie group  $G$  and a principal  $G$ -bundle such that the total space has the homotopy type of  $P$  and the associated Forgetable map  $F$  has uncountable kernel?*

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